

# Anisotropy and scaling corrections in turbulence

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Two parametrizations for second order velocity moments, the Batchelor parametrization for the r-space structure function and a common parametrization for the energy spectrum,  $E(p) \propto p^{-5/3} \exp(-p/p_d)$ , are examined and compared. In particular, we investigate corrections to the local scaling exponents induced by finite size effects. The behavior of local r- and p-space exponents differs dramatically. The Batchelor type parametrization leads to energy pileups in p-space at the ends of the ISR. These bottleneck effects result in an extended r-space scaling range, comparable to experimental ones for the same Taylor-Reynolds number  $Re_\lambda$ . Shear effects are discussed in terms of (global) apparent scaling correction  $\delta\zeta^{app}$  to classical scaling. The scaling properties of  $\delta\zeta^{app}(Re_\lambda)$  differ not only among the parametrizations considered, but also among r- and p-space for a *given* parametrization. The difference can be traced back to the subtleties of the crossovers in the velocity moments. Our observations emphasize the need for more experimental information on crossovers between different subranges.

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# 1 Introduction

In the theory of fully developed turbulence, scaling ranges of velocity moments in  $r$ - and in  $p$ -space are often put into a one-to-one correspondence with each other. The two scaling exponents associated with the scaling ranges are believed to be equivalent. While this view is correct for an infinite system (with an infinite scaling range) the relation between  $r$ -space and  $p$ -space exponents becomes more complicated for finite Reynolds numbers. It is the aim of this paper to quantitatively examine these finite size effects, as they might well be essential to interpret experiments [1, 2, 3] and numerical simulations [4, 5].

We begin our short review of previous work relevant to the present investigation by defining the *infinite scaling range* exponents. In  $r$ -space, they are defined via the velocity structure functions

$$D^{(m)}(r) = \langle (\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x}))^m \rangle \propto r^{\zeta_m}. \quad (1.1)$$

From a theoretical point of view [6, 7, 8], the  $p$ -space scaling exponents corresponding to the (discrete) Fourier transformation  $\mathbf{u}(\mathbf{p})$  of  $\mathbf{u}(\mathbf{x})$  are more easily accessible,

$$\langle |\mathbf{u}(\mathbf{p})|^m \rangle \propto p^{-\zeta_m}. \quad (1.2)$$

Kolmogorov's classical dimensional analysis of the turbulence problem [9] gives of course the same result for both kinds of scaling exponents, namely  $\zeta_m = m/3$ . Since Landau's famous footnote in ref. [10] it has been a matter of interest whether there are scaling corrections  $\delta\zeta_m = \zeta_m - m/3$  to the classical result in the limit of *infinite* Reynolds number  $Re$  [11, 12, 7, 8].

Grossmann and Lohse discovered and analyzed *finite* size scaling corrections in their reduced wave vector set approximations (REWA, see [13, 14] and refs. therein) of the Navier-Stokes equations. These calculations fill the huge gap in the Taylor-Reynolds number  $Re_\lambda$  between experiments with  $Re_\lambda > 10^4$  [15, 3] and full numerical simulations, which reach only  $Re_\lambda \approx 200$  [5]. Similar to the highest  $Re_\lambda$  experiments, REWA [14, 16] achieves  $Re_\lambda > 10^4$ , thus resolving four decades in  $p$ -space. Three different ranges were distinguished in  $p$ -space: The stirring subrange (SSR, small  $p$ ) with slight intermittency (i.e., scaling corrections), the viscous subrange (VSR, large  $p$ ) with strong intermittency, and the inertial subrange (ISR, medium  $p$ ) with hardly any intermittency [13, 16]. The physical origin of the VSR and SSR scaling corrections was extensively discussed in [13]. The VSR corrections arise from the competition between turbulent energy transfer downscale and viscous damping [17, 18]. The SSR scaling corrections are presumably due to the broken symmetry of the Navier-Stokes dynamics because of the *finite size* of the system: For small  $p$  *only* downscale energy transfer is possible, i.e., the translational invariance and the self similarity of the turbulent flow is broken by the boundary conditions. In addition to the investigation of these local scaling corrections, REWA also offered the opportunity to study the  $Re_\lambda$ -dependence of *global* corrections to classical scaling.

Values of  $Re_\lambda$  ranging from  $10^2$  to  $10^4$  could be simulated [14, 19] and it was shown [19] that  $\delta\zeta_m \propto Re_\lambda^{-3/5}$  due to the spectral corrections to classical scaling,

$$\langle |\mathbf{u}(\mathbf{p})|^m \rangle \propto p^{-m/3} \left( 1 + \alpha_m \left( \frac{p}{p_s} \right)^{-2/3} \right), \quad (1.3)$$

which result from large scale anisotropy (e.g., shear). The  $p^{-2/3}$  shear correction has first been suggested by Lumley [20], who employed dimensional analysis, and was later also found in refs.[21, 22, 19, 23] with the help of dimensional analysis in terms of Clebsch variables. In refs. [21, 24] the correction has been associated with conserved helicity flux in  $p$ -space. The parameter  $p_s$  is the typical scale set by the strength of the shear  $s$ ,  $p_s = \sqrt{s^3/\epsilon}$ , and  $\alpha_m$  a dimensionless parameter, presumably on the order of 1. It is not yet clear, whether the second term in eq.(1.3) is only a small- $p$  correction or whether pure shear energy spectra  $E(p) \propto \langle |\mathbf{u}(\mathbf{p})|^2 \rangle / p \propto p^{-7/3}$  exist, i.e., whether  $p_s \gg p_L$  can be achieved. Here,  $p_L \equiv 1/L$  is the momentum scale set by the external stirring force. If pure shear spectra exist, they will probably be more pronounced in cross spectra [25]  $E_{12}(p)$ . Yakhot [23] recently discussed experimental indications for pure shear spectra. Whether Maloy and Goldberg's [26] scaling of the velocity structure function  $D^{(2)}(r) \propto r^{4/3}$  in Taylor-Couette flow with oscillatory inner cylinder corresponds to (1.3) is also not clear.

A systematic analysis of the properties of  $r$ - and  $p$ -space scaling exponents in finite- $Re_\lambda$  turbulence has been performed by the present authors in ref.[27]. We demonstrated that the  $r$ -space crossover from the ISR to the VSR and from the ISR to a large- $r$  saturation range can lead to energy-pileups at both ends of the  $p$ -space ISR, the so-called bottleneck phenomenon [28]. In other words, monotonous local  $r$ -space scaling exponents may give rise to non-monotonous local  $p$ -space exponents. Both physical (based on the conserved energy current in  $p$ -space) and formal explanations for this effect as well as a comparison with available experimental and numerical data were given in ref. [27].

In the present paper, we continue and extend our investigation of finite size effects on local scaling exponents in  $r$ - and in  $p$ -space. In the absence of analytical techniques, that would enable us to study these questions directly from a dynamical point of view (which would of course be preferable), our strategy is the following: Throughout the paper we will compare two common parametrizations (and finite size corrections thereof) for the scaling behavior of velocity moments, namely the  $p$ -space parametrization (2.1) discussed by Foias, Manely and Sirovich [29] (henceforth called 'FMS-Parametrization' for simplicity) and the Batchelor-parametrization (2.2) [30, 25], common in  $r$ -space. The FMS-parametrization has already been discussed in earlier work by Brachet et al. [31] and Frisch et al. [18]. We will see that the main difference between FMS- and Batchelor-parametrization lies in the description of the crossover from the viscous to the inertial range. In this paper, these parametrizations are considered to be two examples for possible crossover scenarios and we hope that further experimental results will allow for an

unambiguous discrimination between them. We believe that pointing out the importance and the surprising consequences of the nature of the crossovers is a main novel aspect of our work, since up to now research was mainly concerned with infinite  $Re$  scaling exponents.

The paper consists of two major parts: The first one deals with local small- $p$  scaling corrections. The bottleneck energy pileup at the infrared end of the ISR [27] as well as the small  $p$  scaling corrections found in REWA [13] and also by dimensional analysis in terms of Clebsch variables [19, 21, 23] belong to this category. We believe that these are all manifestations of the broken Navier–Stokes symmetry due to the boundaries, i.e., large scale anisotropy [13]. To investigate possible relations among these effects we modify the above-mentioned parametrizations to describe the scaling corrections from REWA, examine the ensuing consequences in  $r$ -space, and perform a quantitative comparison for the local scaling exponents  $\zeta_2(p)$  resulting from the three different approaches in refs.[13, 19, 27].

In the second part of the paper we focus our attention on the Taylor–Reynolds number dependences of the apparent (global) scaling corrections  $\delta\zeta_2^{app}$  due to shear effects. While it is probably not too surprising that Batchelor– and FMS–parametrization lead to different  $Re_\lambda$ -dependences of the apparent scaling corrections  $\delta\zeta_2^{app}$ , our result, that *both* parametrizations give rise to *different* behaviour of  $\delta\zeta_2^{app}$  in momentum and coordinate space, respectively, is certainly unexpected. E.g., we find that the  $p$ -space result  $\delta\zeta_2^{app,p} \propto Re_\lambda^{-3/5}$  of ref.[19, 7] corresponds to  $\delta\zeta_2^{app,r} \propto Re_\lambda^{-1/2}$  in  $r$ -space. This observation can be viewed as yet more evidence for the fact that finite size effects can render  $r$ - and  $p$ -space exponents inequivalent.

Specifically, our paper is organized as follows: In Sect.2 we introduce the FMS– and Batchelor–parametrizations and review our earlier calculations [27] comparing both parametrizations in  $p$ - and in  $r$ -space. Sect.3 contains the corresponding analysis for power spectra and correlation functions in the time domain [32]. In Sect.4 we investigate the consequences of the SSR scaling corrections found in REWA and compare the small- $p$  scaling corrections of refs.[13, 19, 27]. Sect.5 is devoted to a detailed study of shear effects, in particular, to the apparent scaling corrections  $\delta\zeta_2^{app}$  induced in  $r$ - and in  $p$ -space. Finally, Sect.6 contains our summary and conclusions.

## 2 Batchelor– versus FMS–parametrization

### 2.1 Definitions and Fourier transforms

To describe the behavior of energy spectra  $E(p)$ , the FMS–parametrization [29, 18, 31]

$$E_{\text{FMS}}(p) = E_0 \epsilon^{2/3} p^{-5/3} \exp(-p/p_d) \quad (2.1)$$

has frequently been used to interpret experimental [33, 34] and numerical [5, 13, 14] data. Here,  $E_0$  is the  $p$ -space Kolmogorov constant,  $\epsilon$  the energy dissipation rate, and  $p_d$  characterizes the crossover to the viscous range. On the other hand, measured

structure functions  $D^{(2)}(r)$  are well described by the Batchelor parametrization [30, 25, 1, 35, 36, 27, 37],

$$D_B^{(2)}(r) = \frac{\epsilon r^2 / (3\nu)}{\left(1 + \left(\frac{1}{3b}\right)^{3/2} \left(\frac{r}{\eta}\right)^2\right)^{2/3}}, \quad (2.2)$$

where  $\nu$  is the viscosity,  $\eta = \nu^{3/4} / \epsilon^{1/4}$  the Kolmogorov length, and  $b = 27\Gamma(4/3)E_0/5 = 6.0 - 8.4$  the experimentally determined [25]  $r$ -space Kolmogorov constant. The generalisation of both parametrizations to  $\zeta_2 \neq 2/3$  is straightforward and was considered in [27]. The essential aspects of our present work do not depend on the precise value of  $\zeta_2$ . Clearly, both parametrizations neglect the finite size of the system since they do not contain a scale for the external stirring force. Furthermore, the FMS-parametrization eq.(2.1) does not contain any energy pilup (or bottleneck effect) [28, 27]. The velocity structure function for a given energy spectrum can be calculated through the Fourier transformation [25]

$$D^{(2)}(r) = 4 \int_0^\infty E(p) \left(1 - \frac{\sin(pr)}{pr}\right) dp. \quad (2.3)$$

Inverting this equation, i.e., calculating the energy spectrum from a given structure function, requires a short discussion. Let us consider turbulence in a large but finite domain, so that  $E(p) \rightarrow 0$  as  $p \rightarrow 0$ . Then the term involving no trigonometric function on the rhs of eq.(2.3) is finite,  $4 \int_0^\infty E(p) dp \equiv D^{(2)}(\infty) < \infty$ . Physically, this term corresponds to the total energy in the fluid. We can now straightforwardly invert eq.(2.3) to give

$$E(p) = -\frac{1}{2\pi} \int_0^\infty pr \sin(pr) \left[D^{(2)}(r) - D^{(2)}(\infty)\right] dr. \quad (2.4)$$

In the limit of infinite system size  $D^{(2)}(\infty)$  grows beyond all bounds, rendering eq.(2.4) ill-defined at first sight. However, since  $\int_0^\infty pr \sin(pr) dr \propto p\delta'(p)$ , this affects only singular contributions at the origin which we may safely discard. We will therefore always use eq.(2.4) with the understanding that  $D^{(2)}(\infty) = 0$ . Formally this means nothing more but that the Fourier transformation of a function will not change (apart from the  $\delta$ -function) if the function is shifted by a constant. With the help of the transformation equations (2.3) and (2.4) we can calculate the structure function corresponding to the FMS-parametrization (2.1) and the energy spectrum associated with the Batchelor-parametrization (2.2), giving

$$D_{\text{FMS}}^{(2)}(r) = \frac{4E_0\Gamma(-2/3)}{r(5/3)p_d^{5/3}} \left(\frac{5}{3}p_d r - \left(1 + p_d^2 r^2\right)^{5/6} \sin\left(\frac{5}{3} \arctan(p_d r)\right)\right). \quad (2.5)$$

and

$$\begin{aligned} E_B(p) &= -\frac{1}{4\pi} \frac{\epsilon}{3\nu} p r_d' \frac{d^3}{dp^3} \int_{-\infty}^\infty \frac{\exp(ipr_d' x)}{(1+x^2)^{2/3}} dx \\ &= E_0 \epsilon^{2/3} p_d'^{-5/3} A \left[ \frac{2}{3} \tilde{p}^{1/6} K_{11/6}(\tilde{p}) + \tilde{p}^{7/6} K_{5/6}(\tilde{p}) \right], \end{aligned} \quad (2.6)$$

respectively. Here, we introduced some abbreviations for simplicity: The dimensionless constant  $A$  has the value  $A = (9\Gamma(1/3))/(\sqrt{2\pi}2^{2/3}5\Gamma(2/3))$ ,  $\tilde{p} = pr'_d$ , and  $K_\nu$  is the modified Bessel function of third kind [38]. The crossover from the inertial to the viscous range is characterized by  $r_d, p_d$  for the FMS-parametrization and by  $r'_d, p'_d$  for the Batchelor-parametrization. The large- and small- $r$  limits of  $D^{(2)}(r)$  are required to give  $D^{(2)}(r) = b(\epsilon r)^{(2/3)}$  and  $D^{(2)}(r) = \epsilon r^2/(3\nu)$ , respectively. Eq.(2.2) is obviously designed to meet these constraints and comparing the asymptotic relations with the appropriate limits of eq.(2.5) fixes  $E_0 = 5b(\Gamma(4/3))^{-1}/27 = 1.74$  and

$$p_d^{-1} = (10b/27)^{3/4}\eta \approx 2.34\eta, \quad (2.7)$$

where we chose  $b = 8.4$  [25]. Now,  $r_d$  and  $r'_d$  are defined by equating the asymptotic limits,  $\epsilon r^2/(3\nu) = b(\epsilon r)^{2/3}$ , so that we arrive at

$$r_d = r'_d = (3b)^{3/4}\eta \approx 11.25\eta. \quad (2.8)$$

We note, however, that although  $r_d$  and  $r'_d$  are the same (by definition),  $D_B^{(2)}(r)$  shows a sharper crossover from VSR to ISR than  $D_{\text{FMS}}^{(2)}(r)$ , i.e.  $D_B^{(2)}(r) \geq D_{\text{FMS}}^{(2)}(r)$  for all  $r$ . This can be seen in Fig.1 of ref.[27] where we compared  $D_B^{(2)}(r)$  with  $D_{\text{FMS}}^{(2)}(r)$ . Finally, the  $p$ -space crossovers  $p_d$  and  $p'_d$  are defined by the cutoff in the exponential decay of the spectrum for large  $p$ . Thus,  $p_d$  is determined by eq.(2.1) and, since  $K_\nu(\tilde{p} = pr'_d) \propto p^{-1/3} \exp(-pr'_d)$  for large argument, we have

$$p_d'^{-1} = r'_d \approx 11.25\eta. \quad (2.9)$$

Note that the naive expectation that (p-space crossover)  $\times$  (r-space crossover)  $\approx 2\pi$ , holds in neither case. For eqs. (2.1) and (2.5) we have  $r_d p_d \approx 4.8$  whereas for eqs. (2.2) and (2.6) we have simply  $r'_d p'_d = 1$ .

## 2.2 Bottleneck phenomenon

In [27] we showed that in contrast to eq.(2.1) the parametrization eq.(2.2) contains an important physical phenomenon, the bottleneck effect [28]. This becomes apparent when comparing the energy spectra  $E_{\text{FMS}}(p)$  and  $E_B(p)$  in Fig.1. For small  $p$  both functions coincide. Around  $p \approx p'_d$ , however, an energy pileup in the crossover region of  $E_B(p)$  becomes noticable, leading to a non-monotonous logarithmic slope  $d \log E_B / d \log p$ . The local r- and p-space scaling exponents

$$\zeta_2(r) = \frac{d \log D^{(2)}(r)}{d \log r}, \quad -\zeta_2(p) - 1 = \frac{d \log E(p)}{d \log p} \quad (2.10)$$

of both the Batchelor- and the FMS-parametrization are plotted in the right part of Fig.2. The minimal local p-space scaling exponent of the spectrum  $E_B(p)$  is 0.44, i.e., the scaling correction is an order of magnitude larger than the discussed intermittency corrections [25, 11]. This effect could explain, that the spectra in

numerical simulations [4, 5, 39] are *flatter* than the classical expectation, rather than being steeper as one might expect from possible intermittency corrections. To quantify this effect, we have introduced an *effective* (global) p-space scaling exponent  $\zeta^{eff}(Re_\lambda)$  in ref. [27] which only slowly approaches its infinite  $Re_\lambda$  value. Very recent measurements of  $\zeta^{eff}(Re_\lambda)$  by Zocchi et al. [40] showed exactly the same behavior for  $\zeta^{eff}(Re_\lambda)$  [27].

Furthermore, in ref.[27] we considered a straightforward generalization of the Batchelor-parametrization,

$$D_B^{(2)}(r) \propto r^2 \cdot (r_d'^2 + r^2)^{-2/3} \cdot (L^2 + r^2)^{1/3}, \quad (2.11)$$

which accounts for the crossover from the ISR to a large- $r$  saturation range induced by the finite scale  $L = 1/p_L$  set by the external stirring force. This second crossover might well not be universal, but our parametrization agrees reasonably well with available data [1, 25, 36]. Calculating the corresponding spectrum we obtained for  $r_d' \ll r$

$$\begin{aligned} E_B(p) = & \frac{\langle \mathbf{u}^2 \rangle L}{\pi} \left( -\frac{\Gamma(5/6)}{\Gamma(1/3)} \sqrt{\pi} \left[ \frac{5}{9} \bar{p}^2 {}_1F_2\left(\frac{11}{6}, \frac{5}{2}, \frac{5}{2}, \frac{\bar{p}^2}{4}\right) + \frac{11}{405} \bar{p}^4 {}_1F_2\left(\frac{17}{6}, \frac{7}{2}, \frac{7}{2}, \frac{\bar{p}^2}{4}\right) \right] \right. \\ & \left. + \frac{\pi}{2} \left[ \frac{1}{3} \bar{p} {}_1F_2\left(\frac{4}{3}, 2, \frac{3}{2}, \frac{\bar{p}^2}{4}\right) + \frac{2}{27} \bar{p}^3 {}_1F_2\left(\frac{7}{3}, 3, \frac{5}{2}, \frac{\bar{p}^2}{4}\right) \right] \right) \end{aligned} \quad (2.12)$$

where  $\bar{p} = p/p_L$  and  ${}_1F_2(a, b, c, z)$  denotes a generalized hypergeometric function [41]. The most prominent feature of this expression is a second bottleneck pileup at the infrared end of the p-space ISR. The local logarithmic slopes of (2.11) and (2.12) are plotted in the left part of Fig.2. Both bottlenecks have the same physical origin, namely the broken symmetry due to finite size effects. The symmetry breaking scale is introduced by the stirring force and the finite size of the vessel, wind channel, or atmosphere [1, 3] at the infrared end of the spectrum, and by viscosity at the large- $p$  end of the ISR. Formally, both bottleneck energy pileups originate from the sharp r-space crossovers defined by the Batchelor-parametrization eq.(2.11). The physical explanation [14, 27] builds on the constant energy flux  $T(p) \sim pu(\mathbf{p}) \int d\mathbf{p}_1 d\mathbf{p}_2 u(\mathbf{p}_1) u(\mathbf{p}_2) \delta(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2)$  downscale in p-space. For a detailed discussion of the bottleneck effect we refer to ref.[27].

## 2.3 Higher order moments

The preceding analysis cannot easily be extended to higher order structure functions. The connection between, say, the fourth order structure function  $D^{(4)}(r)$  and the corresponding fourth moment of  $\mathbf{u}(\mathbf{p})$ ,

$$\begin{aligned} D^{(4)}(r) \propto & \int \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \langle \mathbf{u}(\mathbf{p}_1) \mathbf{u}(\mathbf{p}_2) \mathbf{u}(\mathbf{p}_3) \mathbf{u}(\mathbf{p}_4) \rangle \\ & \langle \prod_{j=1}^4 (\exp(i\mathbf{p}_j \cdot \mathbf{r}) - 1) \rangle_{\text{angle}} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4, \end{aligned} \quad (2.13)$$

is considerably more complicated than eq.(2.3). Therefore we have to restrict ourselves to a few general remarks.

Neglecting intermittency corrections we assume as a first approximation that  $\mathbf{v}_r(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)$  and  $\mathbf{u}(\mathbf{p}, t)$  are Gaussian distributed, so that we may simply factorize higher moments (for even  $m$ ),

$$\begin{aligned} \langle |\mathbf{u}(\mathbf{p})|^m \rangle &\propto \langle |\mathbf{u}(\mathbf{p})|^2 \rangle^{m/2}, \\ D^{(m)}(r) &\propto \left( D^{(2)}(r) \right)^{m/2}. \end{aligned} \quad (2.14)$$

Note, however, that the above assumptions are not independent. In a completely homogeneous medium the second moments in p-space are local (i.e.,  $\langle \mathbf{u}^*(\mathbf{p})\mathbf{u}(\mathbf{p}') \rangle \propto \delta(\mathbf{p} - \mathbf{p}')$ ) and the second line in eq.(2.14) is a direct consequence of the first one. Of course, the assumption of Gaussian factorization in eq.(2.14) is at variance with the fact that odd moments do not vanish (e.g.,  $D^{(3)}(r) < 0$  for large  $r$  due to Kolmogorov's structure equation [25]).

The similar looking factorizations in eq.(2.14) lead to quite different results concerning the  $m$ -dependence of the crossovers  $r_d'^{(m)}$  and  $p_d^{(m)}$  (or  $p_d'^{(m)}$ ) between VSR and ISR. These lengths are defined as above by matching the asymptotic behavior for large and for small  $r$  and by the cutoff in the exponentials, respectively. In r-space we get

$$r_d'^{(m)} = r_d'^{(2)} = r_d' = \text{constant} \quad (2.15)$$

for all  $m$ , which is in agreement with recent measurements [37], while in p-space

$$p_d^{(m)} = 2p_d^{(2)}/m = 2p_d/m \quad (2.16)$$

becomes smaller with increasing  $m$ . (The same relation holds for  $p_d'$ ). Eq. (2.16) has been numerically confirmed to a high precision [13]. Thus, for increasing  $m$  the ISR becomes smaller and smaller in p-space, whereas it remains invariant in r-space. Technically, this is due to the fact that we compare two power laws in r-space and a power law with an *exponential* in p-space. An intuitive understanding, we hope, is provided by the following remark: Raising  $D^{(2)}(r)$  to some power smoothes the transition from VSR to ISR and consequently reduces the corresponding spectral strength at large values of  $p$ .

### 3 Frequency spectra

Instead of performing our analysis for the structure function  $D^{(2)}(r) = D(r)$  (for simplicity, we drop the index 2) and the spectrum  $E(p)$  we can do the same for the *longitudinal* structure function  $D_L^{(2)}(r) = D_L(r)$  and the *longitudinal* spectrum  $E_1(p)$  [25], which are connected with the experimentally most easily accessible time structure function  $D(\tau) = \langle (u_1(t + \tau) - u_1(t))^2 \rangle$  and its frequency power spectrum

$$P(\omega) = -\frac{1}{\pi} \int_0^\infty d\tau \cos(\omega\tau) D(\tau) \quad (3.1)$$



by Taylor's hypothesis [42, 25].

As Batchelor's parametrization is also an excellent fit to the directly measured time structure function  $D(\tau)$  [25, 36, 27], we want to give the corresponding frequency spectrum  $P(\omega)$  for completeness. Again we restrict ourselves to classical scaling.

First, for the VSR-ISR crossover

$$D(\tau) \propto \frac{\tau^2}{(1 + (\tau/\tau_d)^2)^{2/3}}, \quad (3.2)$$

we obtain from eq. (3.1)

$$P(\omega) \propto -\bar{\omega}^{-5/6} K_{5/6}(\bar{\omega}) + \bar{\omega}^{1/6} K_{11/6}(\bar{\omega}). \quad (3.3)$$

Here,  $\tau_d$  characterizes the VSR-ISR crossover and  $\bar{\omega} = \omega\tau_d$ . The local logarithmic slopes  $\zeta(\tau)$  and  $\zeta(\omega)$  of (3.2) and (3.3) (defined as in (2.10)) are plotted in the right part of Fig.3. Similar to  $\zeta(p)$ , the local slope  $\zeta(\omega)$  is again non-monotonous, reflecting the bottleneck phenomenon. But for  $P(\omega)$  (and thus  $E_1(p)$ ) it is only half as large as above: The local exponent  $\zeta(\omega)$  has a minimum of 0.56, compared to 0.44 of  $\zeta(p)$ , found in [27], see also Fig. 2.

Next, for the ISR-large  $\tau$  saturation range we assume (again led by experimental data [1, 35, 25]),

$$D(\tau) \propto \tau^{2/3} \left(1 + (\tau/\tau_L)^2\right)^{-1/3}. \quad (3.4)$$

This second crossover is governed by the large eddy turnover time-scale  $\tau_L$ . The corresponding spectrum reads

$$P(\omega) \propto \frac{\Gamma(5/6)}{\sqrt{\pi}\Gamma(1/3)} \left( {}_1F_2\left(\frac{5}{6}, \frac{3}{2}, \frac{3}{2}, \tilde{\omega}^2\right) + \frac{20}{27}\tilde{\omega}^2 {}_1F_2\left(\frac{11}{6}, \frac{5}{2}, \frac{5}{2}, \tilde{\omega}^2\right) \right) - \frac{1}{3}\tilde{\omega} {}_1F_2\left(\frac{4}{3}, 2, \frac{3}{2}, \tilde{\omega}^2\right), \quad (3.5)$$

where  $\tilde{\omega} = \omega\tau_L/2$ . The local logarithmic slopes of (3.4) and (3.5) are plotted in the left part of Fig.3. The maximal value of  $\zeta(\omega)$  is 0.73 instead of 0.77 found for  $\zeta(p)$  in [27]. Again, the bottleneck effect is smaller in  $\omega$ -space than in  $p$ -space.

Our Fig.3 has to be compared with the experimental local slope of  $P(\omega)$ , which we show in Fig.4, taken from Praskovsky and Oncley's recent paper [3]. While the comparison is definitely not conclusive due to the experimental noise, there nevertheless seems to be a certain tendency towards the formation of energy pilups at both ends of the ISR scaling range. For a discussion of the quantitative discrepancy we refer to section 6.

## 4 Small- $p$ scaling corrections

In this section we examine the consequences of the  $p$ -space SSR scaling corrections found by Grossmann and Lohse [13, 14] in their numerical analysis. To this end

we modify the energy spectra  $E_B(p)$  and  $E_{FMS}(p)$  calculated in section 2 to include these corrections at the infrared end of the ISR. Two major problems arise in the course of this procedure: First, we have to introduce the external stirring force scale into the  $p$ -space parametrizations to define a finite range for the scaling corrections. Second, we have to calculate the structure functions corresponding to the modified, more complicated energy spectra. In the following we explain how we deal with these problems, present our results, and compare our findings with other small- $p$  corrections [19, 27] that are being discussed.

## 4.1 Infrared cutoff

Apart from our discussion of the small- $p$  bottleneck we have considered only ideal turbulence in an infinite spatial domain up to now. Real turbulence is restricted to a finite range, i.e., there is a maximal length scale  $L$  or, equivalently, a small wave vector cutoff  $p_L \equiv 1/L$ . In order to discuss the small- $p$  bottleneck in section 2 the Batchelor parametrization was generalized to include this length scale. Here, we have to modify the parametrizations  $E_B(p)$  and  $E_{FMS}(p)$  of the energy spectra accordingly. Therefore we multiply the spectra by  $(2/\pi) \arctan((p/p_L)^{11/3})$ . This amounts to imposing energy equipartition  $E(p) \propto p^2$  on the unforced wave vector modes [25, 43, 12] with  $p \ll p_L$ . Note, however, that the small- $p$  behavior of the spectrum in Navier–Stokes dynamics has not been firmly established up to now [12] and we adopt energy equipartition for small  $p$  only as one of several possible scenarios. We will come back to this point at the end of section 4.2. As an immediate consequence of the cutoff the corresponding structure functions saturate for large  $r$  at the constant value  $D^{(2)}(\infty) \approx D^{(2)}(L)$ .

The finite maximal length scale  $L$  allows for the introduction of the Reynolds number  $Re$  or alternatively the Taylor-Reynolds number  $Re_\lambda = \lambda u_{1,rms}/\nu$ , where  $\lambda = u_{1,rms}/(\partial_1 u_1)_{rms}$  is the Taylor length and  $\nu$  the viscosity. Let us express  $p_d$  and  $p'_d$  in terms of  $L$  and  $Re_\lambda$ . We have  $\epsilon = c_\epsilon u_{1,rms}^3/L$  with  $c_\epsilon \approx 1$  known from grid turbulence experiments [44]. Here, we neglect the  $Re_\lambda$ -dependence of  $c_\epsilon$  for small  $Re_\lambda$  [45]. Note, that  $c_\epsilon$  can directly be connected with the Kolmogorov constant  $b$  [45, 27, 46]. On the other hand  $\epsilon = 15\nu(\partial_1 u_1)_{rms}^2$  [25]. Using these relations we finally get  $\eta = 15^{3/4} c_\epsilon^{-1} L Re_\lambda^{-3/2}$  or, with  $c_\epsilon \approx 1$  and the relations (2.7) and (2.9) for  $p_d, p'_d$ ,

$$p_d^{-1} \approx 18 L Re_\lambda^{-3/2}, \quad (4.1)$$

$$p'_d{}^{-1} \approx 86 L Re_\lambda^{-3/2}. \quad (4.2)$$

This establishes the connection between the length scales  $r_d$  (or  $r'_d$ ) and  $L$ , and the Taylor–Reynolds number  $Re_\lambda$ .

## 4.2 REWA scaling corrections and structure functions

In the reduced wave vector set approximations of the Navier-Stokes equation, Grossmann and Lohse [13] found deviations  $\delta\zeta_m(p)$  from classical scaling when *locally* fitting the spectra. These deviations occur – as already discussed in the introduction – only for small  $p$  (SSR intermittency) and for large  $p$  (VSR intermittency), whereas no scaling corrections were found in the p-space ISR [13]. To investigate how the p-space scaling corrections act in r-space, we model a spectrum according to the numerical results in [13] and numerically Fourier transform it into r-space. We will focus attention on p-space SSR scaling corrections. A short calculation reveals that for any function  $E(p)$  with  $-\zeta(p) - 1 = d \log E / d \log p$  a modification defined by

$$\tilde{E}(p) = E(p) \frac{(p_b^\beta + p^\beta)^{\delta/\beta}}{p^\delta} \quad (4.3)$$

leads to a local exponent of the type

$$\tilde{\zeta}(p) = \zeta(p) + \frac{\delta}{1 + (p/p_b)^\beta}. \quad (4.4)$$

This is exactly what we want: Assuming that  $\zeta(p) = 2/3$  we get the modified exponent  $\tilde{\zeta}(p) = 2/3 + \delta$  for small  $p$  until the infrared cutoff sets in. Furthermore the scheme is very flexible. We can introduce positive and negative corrections  $\delta$  and let them become effective for  $p \ll p_b$  or  $p \gg p_b$  depending on the sign of  $\beta$ . We mention that this procedure, when applied to the exact small- $r$  result  $D(r) \propto r^2$  in order to extend  $D(r)$  to the ISR, immediately leads to the Batchelor parametrization (for  $\beta = -2$  and  $\delta = 4/3$ ).

For our present purposes we choose SSR corrections between  $\delta = 0.0 - 0.04$ , in the range of intermittency corrections discussed in the literature [25, 11, 1]. For large  $p$  we have  $\zeta(p) = 2/3$ , (and for very large  $p$  the exponential cutoff is supposed to become effective). The crossover is determined by  $p_b$ , we choose  $p_b = 10p_L - 15p_L$ , i.e., allowing for about one decade of large scale intermittency corrections in p-space as found in [13]. The parameter  $\beta$  determines the smoothness of the transition, we choose  $\beta = 2$ . The two spectra

$$\tilde{E}_{\text{FMS}}(p) = E_{\text{FMS}}(p) \frac{(p_b^2 + p^2)^{\delta/2}}{p^\delta}, \quad \tilde{E}_{\text{B}}(p) = E_{\text{B}}(p) \frac{(p_b^2 + p^2)^{\delta/2}}{p^\delta} \quad (4.5)$$

were each multiplied by the small  $p$  cutoff  $(2/\pi) \arctan((p/p_L)^{11/3+\delta})$  and then Fourier transformed. The Fourier transformations of  $\tilde{E}_{\text{B}}(p)$  and  $\tilde{E}_{\text{FMS}}(p)$  (including the arctan-cutoff) were performed numerically, employing a routine designed to cope with the strongly oscillating integrand. As the strong oscillations in eq.(2.3) are exponentially damped by the asymptotic behavior of  $\tilde{E}(p)$ , there are no serious numerical difficulties. The local scaling exponents for p- and r-space (defined as in eq. (2.10)) are shown in Fig. 5a and 5b, respectively, for a Taylor-Reynolds number of  $Re_\lambda = 3000$ , which is in the range of typical experiments [1].

Let us first discuss the results for the FMS parametrization. Without any small  $p$  corrections (4.5) we have about one decade of more or less constant  $\zeta(p) \approx 2/3$ , Fig. 5a. The corresponding structure function  $D_{\text{FMS}}(r)$ , however, does not scale, see Fig. 5b, where no scaling range can be observed from the local exponent  $\zeta(r)$ . This demonstrates that the transformation from  $p$ - to  $r$ -space is not completely local: A reasonably well-defined scaling range in  $p$ -space is mapped to a  $r$ -space curve with only very poor scaling properties or, perhaps more appropriately, with no scaling range at all. In the REWA calculations of ref. [16] a very similar behavior of  $\zeta(r)$  has been found, which is not surprising, as the spectra are well parametrized by the FMS parametrization [13, 14, 16]. The poor scaling properties of  $D_{\text{FMS}}(r)$  put, in our opinion, a question mark behind the FMS-parametrization, since experimental data for structure functions for the same  $Re_\lambda$  [1] exhibit much better scaling.

Only by introducing the small  $p$  scaling corrections (4.5) to  $E_{\text{FMS}}(p)$ , i.e., by making the scaling properties *worse* in  $p$ -space (Fig. 5a), we get *improved* (but still poor) scaling of the  $r$ -space structure function (Fig. 5b). As examples, we chose  $\delta = 0.02$ ,  $p_b/p_L = 10$ , which is about what was found in the REWA calculations [13, 14], and  $\delta = 0.04$ ,  $p_b/p_L = 15$ . Our result completely agrees with our findings of section 2: *Non-monotonous*  $p$ -space scaling exponents lead to nicer scaling properties in  $r$ -space, i.e., the small  $p$  scaling correction (4.5) can be interpreted as artificially introduced bottleneck energy pileup on the infrared end of the  $p$ -space ISR.

The same analysis is performed for the Batchelor parametrization. Now, due to the large energy pileup at the ultraviolet end of the  $p$ -space ISR, the small- $r$  scaling properties of  $D_B(r)$  are improved considerably. Yet the  $p$ -space arctan infrared energy cutoff still corrupts  $r$ -space scaling properties for large  $r$ . Again, this effect can be partly compensated by local infrared  $p$ -space corrections of type (4.5). With these corrections,  $D_B(r)$  shows better scaling properties in  $r$ -space (Fig. 5b) which are now comparable to the experimentally realized scaling ranges of about 1.5 – 2 decades for that Taylor-Reynolds number  $Re_\lambda = 3000$  [1].

To summarize: *The simple arctan or exp cutoffs of the  $p$ -space ISR scaling range lead to unrealistically short  $r$ -space scaling ranges. Only the energy pileups at both ends of the  $p$ -space ISR lead to a realistic scaling range of the structure function, if compared with experiment [1]. Batchelor parametrizations of the crossovers (2.11, 2.12) include these energy pileups and give realistic scaling ranges for given  $Re_\lambda$ . Our findings also explain, why the  $r$ -space scaling found in REWA is worse than that in the  $p$ -spectra [16], as the latter is quite well described by the FMS parametrization [13].*<sup>1</sup>

Coming back to section 4.1, we briefly mention that it is only the *type* of the ISR to large- $r$  saturation crossover in (2.11) which determines the very small  $p \rightarrow 0$  behavior for  $E_B(p)$ . For  $\beta = 2$  we obtain  $E_B(p) \propto p$ , for  $\beta = 4$  it is  $E_B(p) \propto p^2$ . It is not clear, which behavior is the more realistic one. For the Euler equations the latter can be proven [43] to be correct and simply reflects energy equipartition,

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<sup>1</sup>The bottleneck pileup found in [14] is quantitatively smaller than that following from eq. (2.6), so that the FMS-parametrization is still an appropriate fit.

but for the Navier-Stokes dynamics the situation might well be different [12], in particular, as the thermal energy is orders of magnitude smaller than the energy of the large scale eddies.

### 4.3 Comparison of infrared scaling corrections

As discussed in the introduction, the large scale anisotropy (boundary, shear) leads to eq. (1.3) by dimensional analysis. We wonder how this result compares with the local slope  $\zeta_2(p)$  resulting from (2.12) and from the REWA calculations [13, 14]. Of course we can compare only the crossover region, as in the original work by Lumley [20] and also in the derivation in [21] the second term ( $\propto p^{-2/3}$ ) in eq.(1.3) is considered to be small.

The comparison for the local slopes  $\zeta_2(p)$  is given in Fig. 6. Clearly, although all curves show  $\delta\zeta_2(p) = \zeta_2(p) - 2/3 > 0$  at the infrared end of the p-space ISR, they do not agree quantitatively. Note however, that no wave vectors smaller than the forcing scale were included in REWA. The presence of such wave vectors leads to the infrared bottleneck effect [27]. It would be interesting to include such wave vectors in full numerical simulations or in REWA type simulations and to examine, whether a bottleneck energy pileup as in (2.12) or in (3.5) will occur. Also note from Fig.6, that the REWA corrections, which we introduced into the energy spectra by means of the parametrization chosen in the preceding subsection, could also be described by eq. (1.3) with  $\alpha_2 \approx 0.1$ . This confirms and justifies the analysis done in ref. [19], see also section 5. The bottleneck corrections are comparable in size with the corrections due to eq. (1.3) (with the arctan cutoff) for  $\alpha_2 = 0.5$ .

The quantitative discrepancies in Fig. 6 should not be too surprising. The small  $p$  spectral shape is far from being universal due to different boundary conditions and different kinds of stirring. Yet we believe that the three kinds of discussed small  $p$  scaling corrections  $\delta\zeta_2(p) > 0$  all have the same origin, namely the broken symmetry of the Navier-Stokes dynamics for small  $p$  and the broken self similarity of the turbulent flow.

### 4.4 Higher order moments

In principle, the analysis of section 4.2. can be repeated for higher order velocity moments. In the REWA calculation [13] it was found, that for large  $Re_\lambda$  higher order moments nearly factorize into second order moments, of course apart from the p-space SSR and VSR intermittency corrections. One could assume such a factorization and parametrize the small  $p$  scaling corrections by

$$\zeta_m(p) = \frac{m}{3} + \frac{\delta_m}{1 + (p/p_b)^\beta}, \quad (4.6)$$

with  $\delta_m < 0$  for  $m > 3$ . We have performed this calculation but refrain from discussing the outcome in detail because nothing essentially new can be learnt. The

results simply confirm the views already developed when dealing with the second moments.

## 5 Shear effects

In this section we investigate the consequences of an extended shear scaling range. We will modify the Batchelor- and the FMS-parametrization in such a way that a crossover to a shear scaling range occurs at the scale of the stirring force  $p_L = 1/L$ . We then proceed to calculate the impact of this modification on scaling exponents in both  $r$ - and  $p$ -space. In particular, we focus attention on the *apparent* scaling correction  $\delta\zeta^{app}(Re_\lambda)$  [19] as a function of the Taylor-Reynolds number  $Re_\lambda$ . This correction is induced by the crossover itself and does not depend on the extension of the shear range.

### 5.1 Shear parametrizations

The generalized FMS-parametrization (cf. eqs.(1.3) and (2.1))

$$\langle |\mathbf{u}(\mathbf{p})|^m \rangle \propto p^{-m/3} \left( 1 + \alpha_m \left( \frac{p}{p_L} \right)^{-2/3} \right) \exp(-p/p_d). \quad (5.1)$$

was shown [19] to lead to an apparent scaling correction, defined by

$$\delta\zeta_m^{app,p} = \min_p(\zeta_m(p)) - m/3, \quad (5.2)$$

where, as usual,  $\zeta_m(p) = -d \log \langle |\mathbf{u}(\mathbf{p})|^m \rangle / d \log p$ . For  $p_L \ll p_d$  it was found that

$$\delta\zeta_m^{app,p} = \text{sign}(\alpha_m) \frac{10}{9} \left( \frac{9mp_L}{8p_d} \right)^{2/5} \alpha_m^{3/5} = c_m Re^{-3/10} = c'_m Re_\lambda^{-3/5}, \quad (5.3)$$

i.e., the apparent scaling corrections vanish with increasing  $Re_\lambda$  with a  $-3/5$  power law for all  $m$ . Eq. (5.3) has been numerically confirmed by reduced wave vector set calculations [19]. The dimensionless constants  $c_m$ ,  $c'_m$ , and  $\alpha_m$  are found to be negative for  $m > 3$  [19]. This means that higher order moments do not factorize into second order moments in agreement with the  $p$ -space SSR scaling corrections [13] found numerically. Note that *if* one assumes factorization of higher order moments in the shear range, shear can *not* account for experimentally observed [1] scaling corrections. The prediction (5.3) [19] is also in agreement with very recent experiments which clearly show a *decrease* of  $\delta\zeta_m^{app,p}$  for increasing  $Re_\lambda$  [47].

The local scaling exponent of eq. (5.1) is shown in Fig. 7a for  $Re_\lambda = 1500$  (which is determined according to eq. (4.1)). The apparent scaling correction  $\delta\zeta_2^{app,p} \approx 0.06$  is very large. Note, however, that according to eq.(5.3)  $\delta\zeta_2^{app,p}$  is proportional to  $\alpha_2^{3/5}$  and will therefore be smaller for  $\alpha_2 < 1$ . The  $Re_\lambda$  dependence is displayed in Fig. 8. For large  $Re_\lambda$  the asymptotic result (5.3) is recovered.

We have mentioned in the introduction that the form of the shear correction in eq.(1.3) is based on dimensional analysis only. One could argue that similar considerations as in [20] can as well be directly applied to  $r$ -space expressions. Assuming the transition to the shear range (with  $r$ -space exponent 4/3) to be of the Batchelor type we are led to the ansatz

$$D_B(r) = \frac{\epsilon}{3\nu} r^2 \frac{r_d'^{4/3}}{(r_d'^2 + r^2)^{2/3}} \frac{r_s^{-2/3}}{(r_s^2 + r^2)^{-1/3}}, \quad (5.4)$$

with  $r_s = p_s^{-1} = \sqrt{\epsilon/s^3}$ . We assume again that shear sets in at the stirring scale, hence  $r_s = L$ . For simplicity, we again dropped the index 2 of the structure function, as we will restrict ourselves to  $D^{(2)}(r)$  from here on. Eq.(5.4) should be viewed as a generalization of the Batchelor-parametrization in complete analogy to the generalized FMS-parametrization eq.(5.1). The local scaling exponent of  $D_B(r)$  is given by

$$\zeta(r) = 2 + \frac{2}{3} \frac{r^2}{L^2 + r^2} - \frac{4}{3} \frac{r^2}{r_d'^2 + r^2}, \quad (5.5)$$

and is shown in Fig.7b. The apparent scaling correction is defined analogously to eq.(5.2),

$$\delta\zeta^{app,r} = \min_r(\zeta(r)) - m/3. \quad (5.6)$$

For the same  $Re_\lambda = 1500$  as above,  $\delta\zeta^{app,r} \approx 0.0028$  is now much smaller than the corresponding value for the generalized FMS parametrization. This reflects the much better scaling properties of the Batchelor parametrization (compared to FMS, cf. fig. 5), which we have extensively discussed in sections 2 and 4. The  $Re_\lambda$  dependence of  $\delta\zeta^{app,r}$  is displayed in Fig. 8. For large  $L \gg r_d'$  (i.e., for large  $Re_\lambda$ ) we obtain

$$\delta\zeta^{app,r} = \sqrt{2} \frac{4}{3} \frac{r_d'}{L} \propto Re^{-3/4} \propto Re_\lambda^{-3/2}. \quad (5.7)$$

Hence, also the asymptotic  $Re_\lambda$  dependence of  $\delta\zeta^{app,r}$  for the Batchelor parametrization is quite different from that of  $\delta\zeta^{app,p}$  for the FMS parametrization. One might argue that this is not very astonishing since we compare two different parametrizations. Let us therefore Fourier transform both parametrizations and reexamine their scaling properties thereafter.

## 5.2 Fourier transforms

Transforming the FMS-parametrization eq.(5.1) with the help of eq.(2.3) to  $r$ -space leads to

$$\begin{aligned} D_{\text{FMS}}(r) \propto & \frac{\Gamma(-2/3)}{(5/3)r p_d^{5/3}} \left( \frac{5}{3} p_d r - (1 + p_d^2 r^2)^{5/6} \sin \left( \frac{5}{3} \arctan(p_d r) \right) \right) \\ & + \frac{\alpha_2 \Gamma(-4/3) p_L^{2/3}}{(7/3)r p_d^{7/3}} \left( \frac{7}{3} p_d r - (1 + p_d^2 r^2)^{7/6} \sin \left( \frac{7}{3} \arctan(p_d r) \right) \right). \end{aligned} \quad (5.8)$$

The local slope of eq.(5.8) for  $Re_\lambda = 1500$ ,  $\alpha_2 = 1$ , is also plotted in Fig. 7a, to compare it with the slope of eq. (5.1). Now  $\delta\zeta^{app,r} \approx 0.11$  is even larger than  $\delta\zeta^{app,p}$  (both for FMS), which is clearly understandable from Fig. 5 because of the even worse scaling properties of the FMS parametrization in r-space (compared to p-space).

What is more surprising is, that now for the *same* (FMS) parametrization the  $Re_\lambda$  dependence of  $\delta\zeta^{app,r}$ , cf. Fig. 8, is *different* from that of  $\delta\zeta^{app,p}$ . For the former we obtain the asymptotic result

$$\delta\zeta^{app,r}(Re_\lambda) \propto Re_\lambda^{-1/2}, \quad (5.9)$$

which is a considerable flatter dependence than (5.3).

Finally, we calculate the spectrum corresponding to the generalized Batchelor parametrization (5.4). For  $r \gg r'_d$  we can derive an analytical result, which we give in appendix A for completeness. In the general case, we have to restrict ourselves to a numerical treatment. The numerical Fourier transformation (2.4) of (5.4) (or of (2.2)) is more delicate than the inverse transformation (2.3) due to the absence of an exponential cutoff. In appendix B we explain how we convert the strongly oscillating integral to a rapidly converging one by means of contour integration techniques [48]

The result for the local p-space slope of the generalized Batchelor parametrization (5.4) is shown in Fig. 7b. Of course it shows the ultraviolet bottleneck energy pileup [28], which we had discussed in detail in ref. [27]. But now, in addition, the spectrum shows *reduced* spectral strength at the infrared end of the p-space ISR, i.e., a decreased local slope  $\zeta(p) < 2/3$ . This effect can be interpreted as, so to say, an *inverse* bottleneck effect and can both formally and physically be interpreted along the same line of arguments as the bottleneck pileups discussed above and in [27]. Formally it reflects the sharp crossover from  $r^{2/3}$  to  $r^{4/3}$  scaling in the structure function. Physically [14, 27], the constant energy flux  $T(p) \sim pu(\mathbf{p}) \int dp_1 dp_2 u(\mathbf{p}_1)u(\mathbf{p}_2)\delta(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2)$  downscale now requires *reduced* spectral strength at the infrared end of the ISR, as the spectral strength is increased in the shear range. So it is just the opposite situation as that discussed in section 2.2, see also lhs of Fig. 1 and ref. [27]. Correspondingly, there is also an energy pileup at the high p end of the shear range, which may be a consequence of the constant helicity flux in this region [24]. It leads to a local slope  $\zeta(p) > 4/3$ .

It is not our primary goal to speculate about the nature of this crossover. What is important here for the discussion of apparent scaling corrections is, that *positive* local scaling corrections  $\delta\zeta(r) > 0$  in r-space lead to *negative*  $\delta\zeta(p) < 0$  in the p-space ISR, see Fig. 7b. Thus the apparent scaling corrections  $\delta\zeta^{app,p}$  have to be defined as

$$\delta\zeta_2^{app,p} = \max_{p_L \ll p \ll p_d} (\zeta_2(p)) - m/3, \quad (5.10)$$

The  $Re_\lambda$ -dependence of  $\delta\zeta^{app,p}$  is shown in Fig.8. From the data we conclude that

$$\delta\zeta^{app,p} \propto -Re_\lambda^{-3/2} < 0. \quad (5.11)$$



For  $Re_\lambda = 1500$  we have  $\delta\zeta^{app,p} = -0.0082$ . This means that the qualitative difference between the apparent scaling corrections in  $r$ - and in  $p$ -space is even larger for the Batchelor- than already for the FMS-parametrization. For the former, even the sign of the corrections is reversed as we go from  $r$ - to  $p$ -space, while for the latter only the magnitude and the asymptotic  $Re_\lambda$  scaling exponent change.

Let us summarize the analysis of this section. We found that the subtleties of the ISR to VSR and ISR to shear range crossovers govern the  $Re_\lambda$  dependence of  $\delta\zeta^{app}$ . Moreover, for both parametrizations discussed (generalized FMS and Batchelor) the scaling corrections are quite different in  $r$ - and in  $p$ -space. So, when analyzing the experimental  $Re_\lambda$ -dependence of  $\delta\zeta^{app}$ , one should also expect *different* results in  $r$ - and in  $p$ -space, or, correspondingly, in the  $\tau$ - and  $\omega$ - domain.

Note, that because of the considerable lack of experimental information about the shear range crossover, we consider eqs. (5.1) and (5.4) only as *examples*. As pointed out above, dimensional analysis cannot distinguish between them. We think it is worth while to experimentally or numerically study the crossover between ISR and shear range and hope that our analyses stimulate to do so. This might lead to a better understanding of the  $Re_\lambda$  dependence of scaling corrections.

## 6 Summary and conclusions

Throughout the paper we have demonstrated, that scaling properties in  $r$ - and in  $p$ -space can be quite different. One could argue that in the infinite  $Re_\lambda$  limit these differences are irrelevant. This is of course correct. Yet, as we demonstrated, for those  $Re_\lambda$  which can be achieved in experiments and even more so for the numerical ones, the finite size corrections are considerable and it is important to know what their influence is to be able to interpret the data correctly. Moreover, the apparent scaling correction  $\delta\zeta^{app}$  due to shear corrections even show *asymptotically* different  $Re_\lambda$  scaling behavior. For the FMS type parametrization we had obtained  $\delta\zeta^{app,p} \propto Re_\lambda^{-3/5}$  [19, 7] and  $\delta\zeta^{app,r} \propto Re_\lambda^{-1/2}$ , for Batchelor type parametrizations  $\delta\zeta^{app,r} \propto Re_\lambda^{-3/2}$  and  $-\delta\zeta^{app,p} \propto Re_\lambda^{-3/2}$ .

Comparison of the size of the scaling ranges for given  $Re_\lambda$  between experiment [1] and our parametrizations lets us favor a Batchelor type parametrization rather than a parametrization of FMS type. The latter, consisting of a power law in  $p$ -space with a large- $p$  exponential cutoff and a small- $p$  arctan cutoff, does not exhibit any bottleneck energy pileups at the ends of the  $p$ -space ISR and leads to unrealistic short scaling ranges in the  $r$ -space structure function. In other words, combining all regimes discussed in this paper (VSR, ISR, shear range and large- $r$  saturation range) we think that the  $p$ -space parametrization

$$E(p) = \frac{2E_0\epsilon^{2/3}}{\pi} \arctan \left( \left( \frac{p}{p_L} \right)^{11/3} \right) p^{-5/3} \left( 1 + \left( \frac{p}{p_s} \right)^{-2/3} \right) \exp(-p/p_d), \quad (6.1)$$

with  $p_d \geq p_s \geq p_L$  is less favorable than an  $r$ -space parametrization

$$D(r) = \frac{\epsilon}{3\nu} r^2 \frac{r_d'^{4/3}}{(r_d'^2 + r^2)^{2/3}} \frac{r_s^{-2/3}}{(r_s^2 + r^2)^{-1/3}} \frac{L^{-4/3}}{(L^2 + r^2)^{-2/3}}, \quad (6.2)$$

with  $r_d' \leq r_s \leq L$ , which shows bottleneck effects in  $p$ -space. In many isotropic turbulence experiments  $r_s \approx L$  and the shear range will be suppressed. If less isotropy is achieved in experiments, we may have, say,  $r_s \approx L/4$ . In this case, the energy pileup due to large- $r$  saturation (section 2 and ref. [27]) and the spectral strength reduction due to shear effects will partly compensate each other at the infrared end of the  $p$ -space ISR. This leads to a smaller change of the local slope than predicted by eq. (2.12) or (3.5).

Another point to be kept in mind when comparing our predictions (2.12) and (3.5) with experimental data is the issue of averaging. While experimental scaling exponents necessarily represent data averaged over a certain interval, we defined a *pointwise* (local) slope in (2.10) and used this basic quantity throughout the paper. In the numerical REWA calculations [13, 14], *locally averaged* instead of pointwise slopes were determined by fitting the parametrization (2.1) to the data in each interval  $[p/\sqrt{10}, p\sqrt{10}]$ . To estimate the effect of averaging, we performed a running average of the pointwise slopes in fig.3 (the quantity most easily accessible in experiments) using an averaging range of  $[\omega/\sqrt{10}, \omega\sqrt{10}]$ , see fig. 9. As expected, the bottleneck pileups become attenuated, so that they are now quantitatively closer to the measured ones in fig.4. Similar averaging algorithms could also be applied to the other local slopes.

Finally, we mention that while eqs. (6.1) and (6.2) explicitly distinguish between shear effects and large  $r$  saturation effects, caused by the boundary conditions (i.e., large scale anisotropy), these effects might be more interwoven. In section 4.3 we had compared shear effects, finite size effects, and the numerical REWA results. Qualitatively, they all lead to  $\delta\zeta(p) > 0$  for small  $p$ , but the quantitative agreement was less satisfactory. We again stress the necessity to produce as clean shear ranges as possible in experiments. Only then, one will be able to experimentally study the crossover phenomena associated with the shear range, which – together with those from VSR to ISR – might well be a key in understanding scaling corrections.

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## A Analytical Batchelor–type shear spectrum

We can analytically perform the transformation of the Batchelor–parametrization eq.(5.4) under the assumption that  $r \gg r'_d$ , so that

$$D(r) \approx \frac{\epsilon}{3\nu} \left( \frac{r}{r_s} \right)^{2/3} \frac{1}{(r_s^2 + r^2)^{-1/3}}. \quad (\text{A.1})$$

With this approximation we get

$$\begin{aligned} E(p) &= -\frac{1}{2\pi} \int_0^\infty pr D^{(2)}(r) \sin(pr) dr \\ &= \frac{p}{2\pi} \frac{\epsilon}{3\nu r_s^{2/3}} \frac{d}{dp} \left( r_s^2 - \frac{d^2}{dp^2} \right) \int_0^\infty r^{2/3} (r_s^2 + r^2)^{-2/3} \cos(pr) dr. \end{aligned} \quad (\text{A.2})$$

The integral in eq.(A.2) can be solved [41] to give

$$\begin{aligned} &\int_0^\infty r^{2/3} (r_s^2 + r^2)^{-2/3} \cos(pr) dr \\ &= -3r_s^{1/3} \frac{\Gamma(5/6)^2}{\Gamma(2/3)} {}_1F_2\left(\frac{5}{6}; \frac{1}{2}, \frac{7}{6}; \frac{\tilde{p}^2}{4}\right) + \frac{\sqrt{3}}{2} \Gamma(1/3) p^{-1/3} {}_1F_2\left(\frac{2}{3}; \frac{1}{3}, \frac{5}{6}; \frac{\tilde{p}^2}{4}\right), \end{aligned} \quad (\text{A.3})$$

where  $\tilde{p} = pr_s$ . Reinserting this result into eq.(A.2) we derive after some algebra the following, rather clumsy expression,

$$\begin{aligned} E(p) &= \frac{\epsilon r_s^{5/3}}{3\nu} \left\{ \frac{15}{14\pi} \frac{\Gamma(5/6)^2}{\Gamma(2/3)} \tilde{p}^2 \left[ \frac{11}{13} {}_1F_2\left(\frac{17}{6}; \frac{5}{2}, \frac{19}{6}; \frac{\tilde{p}^2}{4}\right) - {}_1F_2\left(\frac{11}{6}; \frac{3}{2}, \frac{13}{6}; \frac{\tilde{p}^2}{4}\right) \right. \right. \\ &\quad \left. \left. + \frac{187}{3705} {}_1F_2\left(\frac{23}{6}; \frac{7}{2}, \frac{25}{6}; \frac{\tilde{p}^2}{4}\right) \tilde{p}^2 \right] + \right. \\ &\quad \frac{\sqrt{3}}{4\pi} \Gamma(1/3) \tilde{p}^{-7/3} \left[ \frac{28}{27} {}_1F_2\left(\frac{2}{3}; \frac{1}{3}, \frac{5}{6}; \frac{\tilde{p}^2}{4}\right) - \left( \frac{1}{3} {}_1F_2\left(\frac{2}{3}; \frac{1}{3}, \frac{5}{6}; \frac{\tilde{p}^2}{4}\right) + \frac{2}{5} {}_1F_2\left(\frac{5}{3}; \frac{4}{3}, \frac{11}{6}; \frac{\tilde{p}^2}{4}\right) \right) \tilde{p}^2 \right. \\ &\quad \left. \left. + \left( \frac{6}{5} {}_1F_2\left(\frac{5}{3}; \frac{4}{3}, \frac{11}{6}; \frac{\tilde{p}^2}{4}\right) - \frac{9}{11} {}_1F_2\left(\frac{8}{3}; \frac{7}{3}, \frac{17}{6}; \frac{\tilde{p}^2}{4}\right) \right) \tilde{p}^4 - \frac{108}{1309} {}_1F_2\left(\frac{11}{3}; \frac{10}{3}, \frac{23}{6}; \frac{\tilde{p}^2}{4}\right) \tilde{p}^6 \right] \right\}. \end{aligned} \quad (\text{A.4})$$

For small  $p$  the local scaling exponent of  $E(p)$  is  $4/3$ , for large  $p$  it is  $2/3$ , but the transition from one range to the other is non monotonous. This is reflected in the right part of Fig.7b. In the left part of that figure, the ultraviolet bottleneck energy pileup can be seen in addition, which is not included in eqs. (A.1) and (A.4).

## B Contour integration for oscillating integrands

The numerical Fourier transformation (2.4) of (5.4) cannot straightforwardly be performed, as the integrand is strongly oscillating and not exponentially damped. To cope with this problem, we employ contour integration techniques [48]. Plugging (5.4) into (2.4) we obtain after some algebra

$$E(p) = -\frac{\epsilon}{3\nu} r_d'^{4/3} r_s^{-2/3} \frac{p}{4\pi} \frac{d^3}{dp^3} \int_{-\infty}^{\infty} \frac{(r_s^2 + r^2)^{1/3} \exp(ipr)}{(r_d'^2 + r^2)^{2/3}} dr. \quad (\text{B.1})$$

The integral has singularities or zeros at  $\pm ir_d'$  and  $\pm ir_s$ . Taking the correct branch cuts and performing the corresponding contour integration in the upper half plane, we obtain

$$\int_{-\infty}^{\infty} \frac{(r_s^2 + r^2)^{1/3} \exp(ipr)}{(r_d'^2 + r^2)^{2/3}} dr = -\sqrt{3} \int_{r_d'}^{\infty} \frac{|z^2 - r_s^2|^{1/3} \exp(-pz)}{(z^2 - r_d'^2)^{2/3}} dz \quad (\text{B.2})$$

or

$$E(p) = \frac{\epsilon}{3\nu} r_d'^{4/3} r_s^{-2/3} \frac{\sqrt{3}p}{4\pi} \int_{r_d'}^{\infty} \frac{z^3 |z^2 - r_s^2|^{1/3} \exp(-pz)}{(z^2 - r_d'^2)^{2/3}} dz, \quad (\text{B.3})$$

which can now be straightforwardly integrated. Our numerical result is displayed in Fig.7b.

## Figure Captions

Figure 1: In the right part of the main curve we show the energy spectrum eq. (2.6) (solid) with, and the spectrum eq. (2.1) (dashed) without the energy pileup. In the left part the spectrum due to (2.12) is shown. In the insets, the spectrum is enlarged around the energy pileups and compared to classical  $-5/3$ -scaling.

Figure 2: The local  $p$ -space scaling exponents  $\zeta(p)$  (solid), and the local  $r$ -space scaling exponent  $\zeta(r = 1/p)$ .

Figure 3: The local  $\tau$ - and  $\omega$ -space deviation from classical scaling,  $\delta\zeta(\tau = 1/\omega)$  and  $\delta\zeta(\omega)$ , when assuming Batchelor kind crossovers (3.2) and (3.4).

Figure 4: The *experimental* local  $p$ -space deviation from classical scaling,  $\delta\zeta(p)$ , for the longitudinal energy spectrum  $E_1(p)$ . This curve corresponds to the experimental  $\delta\zeta(\omega)$  via the Taylor hypothesis. The data are taken from Praskovsky and Oncley [3] with kind permission of the authors.

Figure 5: (a) Local  $p$ -space scaling exponents  $\zeta(p)$  of the FMS and the Batchelor type energy spectra, both with the arctan cutoff for small  $p$ , see text. To allow for comparison with  $r$ -space, Fig. 5b, we plotted  $\zeta(p)$  versus  $p_L/p$  rather than versus  $p/p_L$ . The Taylor Reynolds number is  $Re_\lambda = 3000$ , cf. eqs. (4.1,4.2). From bottom to top on the rhs of the figure, the three pairs of curves correspond to (i) no small  $p$  scaling corrections, (ii) small  $p$  scaling corrections according to (4.5) with  $\delta = 0.02$  and  $p_b/p_L = 10$ , and (iii) small  $p$  scaling corrections with  $\delta = 0.04$ ,  $p_b/p_L = 15$ . (b) Local  $r$ -space scaling exponents  $\zeta(r)$  for the six curves of Fig. 5a. From bottom to top for both the FMS and Batchelor triple of curves: no scaling corrections,  $\delta = 0.02$ ,  $p_b/p_L = 10$  and  $\delta = 0.04$ ,  $p_b/p_L = 15$ .

Figure 6: Local scaling corrections  $\delta\zeta_2(p) = \zeta_2(p) - 2/3$  due to the REWA calculations [13, 14], due to the infrared bottleneck formula (2.12), and due to eq. (1.3) (with the arctan cutoff for small  $p$ ) with three different values for the unknown parameter  $\alpha_2$ .

Figure 7: (a) Local scaling exponents  $\zeta(r)$  (solid) and  $\zeta(p = \gamma/r)$  (dashed) when shear corrections according to (5.1) are present, with  $Re_\lambda = 1500$  ( $p_d = 3227$ ) and  $\alpha_2 = 1$ . The parameter  $\gamma$  serves to shift the  $p$ -space curve slightly to ensure that the minima of the two curves coincide.

(b) As in (a), but now shear corrections according to (5.4) and its Fourier transform. Again, we chose  $Re_\lambda = 1500$ , ( $p'_d = r_d^{-1} = 676$ ).

Figure 8: Double logarithmic plot of  $\delta\zeta^{app,r}(Re_\lambda)$  and  $\delta\zeta^{app,p}(Re_\lambda)$  for FMS ( $\alpha_2 = 1$ ) and Batchelor parametrization.

Figure 9: Same as in fig. 3, but now in addition the averaged local slopes. The averaging range is  $[\omega/\sqrt{10}, \omega\sqrt{10}]$ .

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